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Abstract

We discuss the recently devised one-loop gap equation for the magnetic mass of hot QCD. An alternative, and one would hope equivalent, gap equation is presented, which however shows no mass generation at the one-loop level.

Submitted to *Physics Letters B*

*This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under contracts #DE-FC02-94ER40818 and #DE-FG02-91ER40676.
BU-HEP-95-31, MIT-CTP-2479 October 1995

In an Abelian plasma, static electric fields are screened (Debye mass or screening length); there is no magnetic screening since there are no magnetic sources. When this problem is treated by thermal quantum field theory, the electric screening mass straightforwardly emerges from Feynman diagrams at high-temperature T, and is found to be of order eT, where e is the coupling strength. In a resummed perturbation expansion, this mass also cures the infrared divergences that afflict un-resummed finite-temperature perturbative expansions when there are massless fields in the theory.

Similar electric mass generation has been demonstrated for non-Abelian gauge theories, but this does not remove all the infrared divergences, which remain when the non-linear interactions of electric (temporal) and magnetic (spatial) degrees of freedom are treated perturbatively [1]. While it is believed that these divergences are also cured by the generation of a magnetic mass μ, a convincing calculation for μ is thus far unavailable. The perturbative resummation, which exposes the Debye mass, gives no evidence for a magnetic mass.

A similar problem arises in three-dimensional (Euclidean) Yang-Mills theory at *zero* temperature, which should provide an effective description for the magnetic (spatial) degrees of freedom of four-dimensional QCD at high-temperature, through the identification of the three dimensional coupling g with $e\sqrt{T}$. Since g^2 carries dimension of mass, it is plausible to suppose that three-dimensional Yang-Mills theory dynamically generates an $O(g^2)$ mass, which eliminates perturbative infrared divergences in the three-dimensional model, and suggests the occurrence of an $O(e^2T)$ magnetic mass in the four-dimensional theory. However, thus far no analysis of the three-dimensional Yang-Mills model has led to a proof of mass generation.

Since the mass is not seen in perturbative expansions, even resummed ones, one attempts a non-perturbative calculation, based on a gap equation. Of course an exact treatment is impossible; one must be satisfied with an approximate gap equation, which effectively sums a large, but still incomplete set of graphs. At the same time, gauge invariance should be maintained; gauge non-invariant approximations are not persuasive.

Deriving an approximate, but gauge invariant gap equation is most efficiently carried out in a functional integral formulation. We begin by reviewing how a one-loop gap equation is gotten from the functional integral, first for a non-gauge theory of a scalar field φ, then we indicate how to extend the procedure when gauge invariance is to be maintained for a gauge field A_μ.

Consider a self-interacting scalar field theory (in the Euclidean formulation) whose potential $V(\varphi)$ has no quadratic term, so in direct perturbation theory one may encounter infrared divergences, and one enquires whether a mass is generated, which would cure them.

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + V(\varphi) \\ V(\varphi) &= \lambda_3\varphi^3 + \lambda_4\varphi^4 + \ldots \end{aligned} \tag{1}$$

The functional integral involves the negative exponential of the action $I = \int \mathcal{L}$. Separating the quadratic, kinetic part of I, and expanding the exponential of the remainder in powers of the field yields the usual loop expansion, which may also be systematized by introducing a loop-counting parameter ℓ and considering $e^{-\frac{1}{\ell}I(\sqrt{\ell}\varphi)}$: the power series in ℓ is the loop expansion. To obtain a gap equation for a possible mass μ, we proceed by adding and subtracting $I_\mu = \frac{\mu^2}{2}\int\varphi^2$, which of course changes nothing.

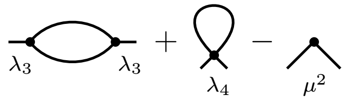
$$I = I + I_\mu - I_\mu \tag{2}$$

Next the loop expansion is reorganized by expanding $I + I_\mu$ in the usual way, but taking $-I_\mu$ as contributing at one loop higher. This is systematized by replacing (2) with an effective action, I_ℓ, containing the loop counting parameter ℓ, which organizes the loop expansion in the indicated manner.

$$I_\ell = \frac{1}{\ell}\Big(I(\sqrt{\ell}\varphi) + I_\mu(\sqrt{\ell}\varphi)\Big) - I_\mu(\sqrt{\ell}\varphi) \tag{3}$$

An expansion in powers of ℓ corresponds to a resummed series; keeping all terms and setting ℓ to unity returns us to the original theory (assuming that rearranging the series does no harm); approximations consist of keeping a finite number of terms: the $O(\ell)$ term involves a single loop.

The gap equation is gotten by considering the self energy Σ of the complete propagator. To one-loop order, the contributing graphs are depicted in Fig. 1.

$$\Sigma = $$ 

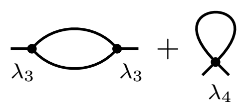
(4)

Fig. 1. Self energy to resummed one-loop order.

Regardless of the form of the exact potential, only the three- and four- point vertices are needed at one-loop order; the bare propagators are massive thanks to the addition of the mass term $\frac{1}{\ell}I_\mu(\sqrt{\ell}\varphi) = \frac{\mu^2}{2}\int\varphi^2$; the last $-\mu^2$ in Fig. 1 comes from the subtraction of the same mass term, but at one-loop order: $-I_\mu(\sqrt{\ell}\varphi) = -\ell\frac{\mu^2}{2}\int\varphi^2$.

The gap equation emerges when it is demanded that Σ does not shift the mass μ. In momentum space, we require

$$\Sigma(p)\Big|_{p^2=-\mu^2} = 0 \tag{5}$$



$$\Big|_{p^2=-\mu^2} = \mu^2$$

Fig. 2. Graphical depiction of Eq. (5).

While the above ideas can be applied to a gauge theory, it is necessary to elaborate them so that gauge invariance is preserved. We shall discuss solely the three-dimensional non-Abelian Yang-Mills model (in Euclidean formulation) as an interesting theory in its own right, and also as a key to the behavior of spatial variables in the physical, four-dimensional model at high temperature.

The starting action I is the usual one for a gauge field.

$$I = \int d^3x \ \mathrm{tr}\ \tfrac{1}{2}F^iF^i$$
$$F^i = \tfrac{1}{2}\epsilon^{ijk}F_{jk} \tag{6}$$

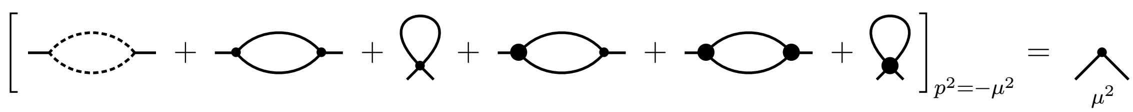
While one may still add and subtract a mass-generating term I_μ, it is necessary to preserve gauge invariance. Thus we seek a gauge invariant functional of A_i, $I_\mu(A)$, whose quadratic portion gives rise to a mass. Evidently

$$I_\mu(A) = -\frac{\mu^2}{2}\int d^3x \ \mathrm{tr}\ A_i\left(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\right)A_j + \ldots \tag{7}$$

The transverse structure of (7) guarantees invariance against Abelian gauge transformations; the question then remains how the quadratic term is to be completed in order that $I_\mu(A)$ be invariant against non-Abelian gauge transformations. [In fact for the one-loop gap equation only terms through $O(A^4)$ are needed.]

A very interesting proposal for $I_\mu(A)$ was given by Nair [1,2] who also put forward the scheme for determining the magnetic mass, which we have been describing. By modifying in various ways the hard thermal loop generating functional (which gives a four-dimensional, gauge invariant but Lorentz non-invariant effective action with a transverse quadratic term), he arrived at a gauge and rotation invariant three-dimensional structure, which can be employed in the derivation of a gap equation.

The scheme proceeds as in the scalar theory, except that $I_\mu(A)$ gives rise not only to a mass term for the free propagator, but also to higher-point interaction vertices. At one loop only the three- and four- point vertices are needed, and to this order the subtracting term uses only the quadratic contribution. Thus the gap equation reads



(8)

Fig. 3. Graphical depiction of Yang-Mills gap equation.

The first three graphs are as in ordinary Yang-Mills theory, with conventional vertices, but massive gauge field propagator (solid line);

$$D_{ij}(p) = \delta_{ij}\frac{1}{p^2+\mu^2} \tag{9}$$

the first graph depicting the gauge compensating "ghost" contribution, has massless ghost propagators (dotted line) and vertices determined by the quantization gauge, conveniently chosen, consistent with (9), to be

$$\mathcal{L}_{\substack{\text{gauge}\\ \text{fixing}}} = \tfrac{1}{2}\boldsymbol{\nabla}\cdot\mathbf{A}(1-\mu^2/\nabla^2)\boldsymbol{\nabla}\cdot\mathbf{A} \tag{10}$$

The remaining three graphs arise from Nair's form for hard thermal loop-inspired $I_\mu(A)$, with solid circles denoting the new vertices. As it happens, the last graph with the four-point vertex vanishes, while the three-point vertex reads

$$
{}^N V^{abc}_{ijk}(\mathbf{p},\mathbf{q},\mathbf{r}) = -\frac{i\mu^2 f^{abc}}{3!(\mathbf{p}\times\mathbf{q})^2}\left\{\frac{1}{3}\left(\frac{\mathbf{p}\cdot\mathbf{q}}{p^2}+\frac{\mathbf{r}\cdot\mathbf{q}}{r^2}\right)p_i p_j p_k - \frac{\mathbf{r}\cdot\mathbf{p}}{3r^2}(q_i q_j p_k + q_i p_j q_k + p_i q_j q_k)\right\} + 5 \text{ permutations} \quad (11)
$$

$$
p+q+r=0
$$

The permutations ensure that the vertex is symmetric under the exchange of any pair of index sets $(a\ i\ p), (b\ j\ q), (c\ k\ r)$. [We discuss the $SU(N)$ theory, with structure constants f^{abc}.]

The result of the computation is

$$
{}^N\Pi^{ab}_{ij} = \delta^{ab}\Pi^N_{ij} \quad (12a)
$$

$$
\Pi^N_{ij} = \Pi^{YM}_{ij} + \overline{\Pi}^N_{ij} \quad (12b)
$$

Π^{YM}_{ij} is the contribution from the first three Yang-Mills graphs and $\overline{\Pi}^N_{ij}$ sums the graphs from $I_\mu(A)$. The reported results are [3]

$$
\begin{aligned}
\Pi^{YM}_{ij}(p) = N(\delta_{ij} - \hat{p}_i\hat{p}_j)&\left[\left(\frac{-13p^2}{64\pi\mu}+\frac{5\mu}{16\pi}\right)\frac{2\mu}{p}\tan^{-1}\frac{p}{2\mu} - \frac{\mu}{16\pi} - \frac{p}{64}\right] \\
&+ N\hat{p}_i\hat{p}_j\left[\left(\frac{p^2}{32\pi\mu}+\frac{\mu}{8\pi}\right)\frac{2\mu}{p}\tan^{-1}\frac{p}{2\mu}+\frac{\mu}{8\pi}-\frac{p}{32}\right]
\end{aligned} \quad (13)
$$

$$
\begin{aligned}
\overline{\Pi}^N_{ij}(p) = N(\delta_{ij} - \hat{p}_i\hat{p}_j)&\left[\left(\frac{3p^2}{64\pi\mu}+\frac{3\mu}{16\pi}\right)\frac{2\mu}{p}\tan^{-1}\frac{p}{2\mu} - \frac{p^2}{8\pi\mu}\left(\frac{\mu^2}{p^2}+1\right)^2\frac{\mu}{p}\tan^{-1}\frac{p}{\mu} + \frac{\mu}{16\pi} + \frac{\mu^3}{8\pi p^2} + \frac{p}{64}\right] \\
&- N\hat{p}_i\hat{p}_j\left[\left(\frac{p^2}{32\pi\mu}+\frac{\mu}{8\pi}\right)\frac{2\mu}{p}\tan^{-1}\frac{p}{2\mu}+\frac{\mu}{8\pi}-\frac{p}{32}\right]
\end{aligned} \quad (14)
$$

The Yang-Mills contribution (13) is not separately gauge-invariant (transverse) owing to the massive gauge propagators. [At $\mu = 0$, Π^{YM}_{ij} reduces to the standard result [4]: $N(\delta_{ij} - \hat{p}_i\hat{p}_j)\left(-\frac{7}{32}p\right)$.] The longitudinal terms in Π^{YM}_{ij} are cancelled by those in $\overline{\Pi}^N_{ij}$, so that the total is transverse.

$$
\Pi^N_{ij}(p) = N(\delta_{ij} - \hat{p}_i\hat{p}_j)\left[\left(\frac{-5p^2}{32\pi\mu}+\frac{1}{2\pi}\mu\right)\frac{2\mu}{p}\tan^{-1}\frac{p}{2\mu} - \frac{p^2}{8\pi\mu}\left(\frac{\mu^2}{p^2}+1\right)^2\frac{\mu}{p}\tan^{-1}\frac{p}{\mu}+\frac{\mu^3}{8\pi p^2}\right] \quad (15)
$$

[Dimensional regularization is used to avoid divergences.]

Before proceeding, let us note the analytic structures in the above expressions, which are presented for Euclidean momenta, but have to be evaluated at the Minkowski value $p^2 = -\mu^2 < 0$. Analytic continuation for the inverse tangent is provided by

$$\frac{1}{x}\tan^{-1}x = \frac{1}{2\sqrt{-x^2}}\ \ln\frac{1+\sqrt{-x^2}}{1-\sqrt{-x^2}} \tag{16}$$

Evidently $\Pi_{ij}^{N}(p)$ possesses threshhold singularities, at various values of $-p^2$.

There is a singularity at $p^2 = -4\mu^2$ (from $\tan^{-1}\frac{p}{2\mu}$) arising because the graphs in Fig. 3, containing massive propagators (9), describe the excahnge of two massive gauge "particles". Moreover, there is singularity at $p^2 = -\mu^2$ (from $\tan^{-1}\frac{p}{\mu}$) and also, separately in Π_{ij}^{YM} and $\overline{\Pi}_{ij}^{N}$, at $p^2 = 0$ (from the $\pm\frac{p}{64}, \pm\frac{p}{32}$ terms). These are understood in the following way. Even though the propagators are massive, the three-point function (11) contains $\frac{1}{p^2}$, $\frac{1}{q^2}$, $\frac{1}{r^2}$ contributions, which act like massless propagators. Thus the threshhold at $p^2 = -\mu^2$ arises from the exchange of a massive line (propagator) together with a massless line (from the vertex). Similarly the threshhold at $p^2 = 0$ arises from the massless lines in the vertex (and also from massless ghost exchange). The expressions acquire an imaginary part when the largest threshhold, $p^2 = 0$, is crossed: Π_{ij}^{YM} and $\overline{\Pi}_{ij}^{N}$ are complex for $p^2 < 0$.

In the complete answer, the $p^2 = 0$ threshholds cancel, and the singularity at the $p^2 = -\mu^2$ threshhold is extinguished by the factor $(\frac{\mu^2}{p^2}+1)^2$. Consequently Π_{ij}^{N} becomes complex only for $p^2 < -\mu^2$, and is real, finite at $p^2 = -\mu^2$.

$$\Pi_{ij}^{N}(p)\Big|_{p^2=-\mu^2} = (\delta_{ij} - \hat{p}_i\hat{p}_j)\,\frac{N\mu}{32\pi}\,(21\ln 3 - 4) \tag{17}$$

From the gap equation in Fig. 3, the result for the mass is [3]

$$\mu = \frac{N}{32\pi}\,(21\ln 3 - 4) \sim \frac{2.384N}{4\pi} \tag{18}$$

[in units of the coupling constant g^2 (or e^2T), which has been scaled to unity].

Before accepting the plausible answer (18) for μ, it is desirable to assess higher order corrections, for example two-loop contributions. Unfortunately, an estimate [3] indicates that 79 graphs have to be evaluated, and the task is formidable.

Here we propose an alternative test for the reliability of the above approach and the stability of the result (18) against corrections.

We suggest deriving the gap equation with a different gauge invariant completion to (6). Rather than taking inspiration from hard thermal loops (which after all have no intrinsic relevance to the three-dimensional gauge theory[1]), we take the following formula for I_μ,

$$I_\mu(A) = \mu^2 \int d^3x \ \mathrm{tr}\ F^i \frac{1}{D^2} F^i \tag{19}$$

where D^2 is the gauge covariant Laplacian. While ultimately there is no *apriori* way to select one gauge-invariant completion to (6) over another, we remark that expressions like (18) appear in two-dimensional gauge theories (Polyakov gravity action, Schwinger model) and are responsible for mass generation. If two- and higher- loop effects are indeed ignorable, this alternative gauge invariant completion, which corresponds to an alternative resummation, should produce an answer close to (18).

With (19), the graphs are again as in Fig. 3, where the propagator is still given by (9) in the gauge (10). However, the three- and four- point vertices in $I_\mu(A)$ are different. One now finds for the three-point vertex

$$V^{abc}_{ijk}(\mathbf{p},\mathbf{q},\mathbf{r}) = \frac{-i\mu^2}{3!} f^{abc} \left(\delta_{ij}\mathbf{q}\cdot\mathbf{r} + q_i p_j\right) \frac{p_k}{p^2 q^2} + \ \text{5 permutations} \tag{20}$$

$$p+q+r=0$$

and the four-point vertex reads

$$V^{abcd}_{ijkl}(\mathbf{p},\mathbf{q},\mathbf{r},\mathbf{s}) = \frac{-\mu^2}{4!} f^{abe} f^{cde} \left\{ \frac{1}{2}\delta_{jk}\,\epsilon_{imn}\,\epsilon_{\ell on}\,\frac{p_m}{p^2}\,\frac{s_0}{s^2} \right. \tag{21}$$
$$\left. -\frac{1}{2r^2}\left(\frac{1}{4}\epsilon_{ijm}\epsilon_{k\ell m} - \epsilon_{imn}\epsilon_{k\ell n}\frac{p_m}{p^2}(p-r-s)_j + \epsilon_{imn}\epsilon_{\ell on}\frac{p_m}{p^2}\frac{s_0}{s^2}(p-r-s)_j(p+q-s)_k\right)\right\}$$
$$+ \ \text{23 permutations}$$

$$p+q+r+s=0$$

With these, the last three graphs in Fig. 3 are evaluated with the help of dimensional regularization, and one finds

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[1]The hot thermal loop generating functional is related to the Chern-Simons eikonal, see Ref. [1]. Since the Chern-Simons term *is* a three-dimensional structure, this fact may provide a basis for establishing the relevance of the hard thermal loop generating functional to three-dimensional Yang Mills theory. The point is under investigation by D. Karabali and V. P. Nair (private communication).

$$\overline{\Pi}_{ij}(p) = N(\delta_{ij} - \hat{p}_i\hat{p}_j)\Bigg(\bigg(\frac{p^6}{128\pi\mu^5} + \frac{p^4}{32\pi\mu^3} + \frac{7p^2}{64\pi\mu} + \frac{27\mu}{64\pi} - \frac{\mu^3}{16\pi p^2}\bigg)\frac{2\mu}{p}\tan^{-1}\frac{p}{2\mu}$$
$$- \bigg(\frac{p^6}{32\pi\mu^5} + \frac{p^4}{16\pi\mu^3} - \frac{p^2}{16\pi\mu} + \frac{\mu}{32\pi}\bigg)\bigg(\frac{\mu^2}{p^2} + 1\bigg)^2 \frac{\mu}{p}\tan^{-1}\frac{p}{\mu}$$
$$- \frac{p^2}{32\pi\mu} - \frac{3\mu}{16\pi} + \frac{49\mu^3}{96\pi p^2} + \frac{\mu^5}{32\pi p^4} + \frac{p^5}{128\mu^4} + \frac{p^3}{32\mu^2} - \frac{p}{16}\Bigg)$$
$$-N\hat{p}_i\hat{p}_j\Bigg(\bigg(\frac{p^2}{32\pi\mu} + \frac{\mu}{8\pi}\bigg)\frac{2\mu}{p}\tan^{-1}\frac{p}{2\mu} + \frac{\mu}{8\pi} - \frac{p}{32}\Bigg) \tag{22}$$

A check on this very lengthy calculation is that summing it with Yang-Mills contribution (13) yields a transverse result.

$$\Pi_{ij}(p) = N(\delta_{ij} - \hat{p}_i\hat{p}_j)\Bigg(\bigg(\frac{p^6}{128\pi\mu^5} + \frac{p^4}{32\pi\mu^3} - \frac{3p^2}{32\pi\mu} + \frac{47\mu}{64\pi} - \frac{\mu^3}{16\pi p^2}\bigg)\frac{2\mu}{p}\tan^{-1}\frac{p}{2\mu}$$
$$- \bigg(\frac{p^6}{32\pi\mu^5} + \frac{p^4}{16\pi\mu^3} - \frac{p^2}{16\pi\mu} + \frac{\mu}{32\pi}\bigg)\bigg(\frac{\mu^2}{p^2} + 1\bigg)^2 \frac{\mu}{p}\tan^{-1}\frac{p}{\mu}$$
$$- \frac{p^2}{32\pi\mu} - \frac{\mu}{4\pi} + \frac{49\mu^3}{96\pi p^2} + \frac{\mu^5}{32\pi p^4} + \frac{p^5}{128\mu^4} + \frac{p^3}{32\mu^2} - \frac{5p}{64}\Bigg) \tag{23}$$

Another check on the powers of $\frac{p}{\mu}$ is that the above reduces to the Yang-Mills result at $\mu = 0$.

Just as (13)–(15), the present formula exhibits threshhold singularities: at $-p^2 = 4\mu^2$, which are beyond our desired evaluation point $-p^2 = \mu^2$; there are also threshhold singularities at $-p^2 = \mu^2$, which are extinguished by the factor $(\frac{\mu^2}{p^2} + 1)^2$; however, those at $p^2 = 0$ do *not* cancel, in contrast to the previous case — indeed $\Pi_{ij}(p)$ diverges at $p^2 = 0$, and is complex for $p^2 < 0$. $\Big[$ It is interesting to remark that the last graph of Fig. 3, involving the four-point vertex, which vanishes in the previous evaluation, here gives a transverse result with unextinguished threshhold singularities at $-p^2 = \mu^2$ and at $p^2 = 0$. The protective factor of $(\frac{\mu^2}{p^2} + 1)^2$ arises when the remaining two graphs are added to form $\overline{\Pi}_{ij}$ of (22), and these also contain non-cancelling $p^2 = 0$ threshhold singularities, as does the Yang-Mills contribution (13) $\Big]$.

Although $\Pi_{ij}(p)\Big|_{p^2=-\mu^2}$ is finite, it is complex and the gap equation has no solution for real μ^2, owing to unprotected threshhold singularities at $p^2 = 0$, which lead to a complex $\Pi_{ij}(p)$ for $p^2 < 0$.

It may be that the hot thermal loop-inspired completion for the mass term (7) is uniquely privileged in avoiding complex values for $-\mu^2 \le p^2 \le 0$, but we see no reason for this.[2] Absent any argument for the disappearance of the threshhold at $p^2 = 0$, and reality in the region $-\mu^2 \le p^2 < 0$, we should expect that also the hot thermal loop-inspired calculation will exhibit such behavior beyond the 1-loop order.[3]

Thus until the status of threshhold singularities is clarified, the self-consistent gap equation for a magnetic mass provides inconclusive evidence for magnetic mass generation. Moreover, if there exist gauge invariant completions for the mass term, other than the hard thermal loop-inspired one, that lead to real Π_{ij} at $p^2 = -\mu^2$, it is unlikely that they all would give the same μ at one loop level, which is further reason why higher orders must be assessed.

[2] We note that the hot thermal loop-inspired vertex (11) is less singular than our (20), when any of the momentum arguments vanish. Correspondingly $\Pi^N_{ij}(p)$ in (15) is finite at $p^2 = 0$, in contrast to $\Pi_{ij}(p)$ which diverges at $\frac{1}{p^2}$. However, we do not recognize that this variety of singularities at $p^2 = 0$ influences reality at $p^2 = -\mu^2$; indeed the individual graphs contributing to Π^N_{ij} are complex at that point, owing to non-divergent threshhold singularities at $p^2 = 0$ that cancel in the sum.

[3] V.P. Nair informs us that at the two loop level, there is evidence for $\ln(1 + \frac{p^2}{\mu^2})$ terms, but it is not known whether they acquire a protective factor of $(\frac{\mu^2}{p^2} + 1)$.

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